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Generalized variational principles for micromorphic magnetoelectroelastodynamics

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ABSTRACT

A family of generalized variational principles is established for the initial–boundary value problem of micromorphic magnetoelectroelastodynamics by He's semi-inverse method. This paper aims at providing a more complete theoretical basis for the finite element applications.

1. Introduction

Magnetoelectroelastic materials are gaining attention because of their capability of converting energies among the magnetic, electric and mechanical forms. Many variational principles have been established in piezoelectricity [1–5], thermopiezoelectricity [6], piezoelectromagnetism [7,8], and magnetoelectroelasticity [9–13]. Recently Nappa [14] and He [15,16] established the variational principles in micromorphic thermoelasticity [17].

In this paper, on the basis of He's contributions [9,16], we will deduce a family of generalized variational principles for the initial–boundary value problem of micromorphic magnetoelectroelastodynamics by He's semi-inverse method [18–21].

2. Fundamental equations

Let Ω be a 3D regular region of an elastic continuum with the piecewise smooth surface $\partial\Omega$. The fundamental equations for the micromorphic behavior [17,14–16] of elastic bodies, coupled with quasi-static electromagnetic fields, consist of the equations of motion

$$\sigma^{ji}_{;j} + F^i = \dot{p}^i, \tag{1}$$

$$m^{kij}_{;k} + \sigma^{ji} - s^{ji} + L^{ij} = \dot{\pi}^{ij}, \tag{2}$$

$$p^i = \rho v^i, \tag{3}$$

$$\pi^{ij} = \rho I^j_k \varpi^{ik}, \tag{4}$$

$$v_i = \dot{u}_i, \tag{5}$$

$$\varpi_{ij} = \dot{\varphi}_{ij}, \tag{6}$$

the constitutive equations

$$\sigma^{ij} = C^{ijmn}_{\varepsilon\varepsilon}\varepsilon_{mn} + C^{ijmn}_{\varepsilon e}e_{mn} + C^{ijmnp}_{\varepsilon\gamma}\gamma_{mnp} - C^{mij}_{\varepsilon E}E_m - C^{mij}_{\varepsilon B}B_m, \tag{7}$$

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$$s^{ij} = C_{\varepsilon e}^{ijmn}\varepsilon_{mn} + C_{ee}^{ijmn}e_{mn} + C_{e\gamma}^{ijmnp}\gamma_{mnp} - C_{eE}^{mij}E_m - C_{eB}^{mij}B_m, \tag{8}$$

$$m^{ijk} = C_{\varepsilon\gamma}^{mnijk}\varepsilon_{mn} + C_{e\gamma}^{mnijk}e_{mn} + C_{\gamma\gamma}^{ijkmnp}\gamma_{mnp} - C_{\gamma E}^{mijk}E_m - C_{\gamma B}^{mijk}B_m, \tag{9}$$

$$D^{i} = C_{\varepsilon E}^{imn}\varepsilon_{mn} + C_{eE}^{imn}e_{mn} + C_{\gamma E}^{imnp}\gamma_{mnp} + C_{EE}^{im}E_m + C_{EB}^{im}B_m, \tag{10}$$

$$H^{i} = C_{\varepsilon B}^{imn}\varepsilon_{mn} + C_{eB}^{imn}e_{mn} + C_{\gamma B}^{imnp}\gamma_{mnp} + C_{EB}^{im}E_m + C_{BB}^{im}B_m, \tag{11}$$

the geometrical equations

$$\varepsilon_{ij} = u_{j;i} - \varphi_{ji}, \tag{12}$$

$$2e_{ij} = \varphi_{ij} + \varphi_{ji}, \tag{13}$$

$$\gamma_{ijk} = \varphi_{ij;k}, \tag{14}$$

the equations for electromagnetic fields

$$\varepsilon^{ijk}H_{k;j} = J^{i}, \tag{15}$$

$$D^{i}_{;i} = q, \tag{16}$$

$$B_i = \varepsilon_{ijk}A^{k;j}, \tag{17}$$

$$E_i = -\phi_{,i}, \tag{18}$$

the boundary conditions

$$u_i = \bar{u}_i \quad \text{on } \partial\Omega_u, \tag{19}$$

$$\sigma^{ij}n_j = \bar{T}^{i} \quad \text{on } \partial\Omega_\sigma, \tag{20}$$

$$\varphi_{ij} = \bar{\varphi}_{ij} \quad \text{on } \partial\Omega_\varphi, \tag{21}$$

$$m^{kij}n_k = \bar{M}^{ij} \quad \text{on } \partial\Omega_m, \tag{22}$$

$$n_iD^{i} = \bar{d} \quad \text{on } \partial\Omega_D, \tag{23}$$

$$\phi = \bar{\phi} \quad \text{on } \partial\Omega_\varphi, \tag{24}$$

$$\varepsilon^{ijk}n_jH_k = \bar{h}^{i} \quad \text{on } \partial\Omega_H, \tag{25}$$

$$A_k = \bar{A}_k \quad \text{on } \partial\Omega_A, \tag{26}$$

and the initial conditions

$$u_{i0} = \bar{u}_{i0}, \tag{27}$$

$$p^{i0} = \bar{p}^{i0}, \tag{28}$$

$$\varphi_{ij0} = \bar{\varphi}_{ij0}, \tag{29}$$

$$\pi^{ij0} = \bar{\pi}^{ij0}. \tag{30}$$

Here σ^{ij} is the stress tensor, s_{ij} is the microstress tensor, m_{kij} is the stress moment tensor, u_i is the displacement vector, υ_i is the velocity vector, p^{i} is the momentum vector, φ_{ij} is the microdeformation tensor, π^{ij} is the micromomentum tensor, ϖ^{ik} is the microvelocity tensor, F_i is the body force vector, L_{ij} is the body moment tensor, ε_{ij}, e_{ij} and γ_{ijk} are the linear strain tensors, ρ is the reference mass density, I_k^{j} is the microinertia, D^{i} is the electric displacement vector, B_i is the magnetic induction vector, E_i is the electric field intensity vector, H^{i} is the magnetic field intensity vector, ϕ is the scalar potential, A_i is the vector potential, q is the electric charge density, J^{i} is the electric current density vector, $C_{XY}^{ij\cdots}$ are the constitutive coefficients, ε^{ijk} is the alternating tensor, ';' denotes covariant differentiation.

3. Variational formulations

Using He's semi-inverse method [18–21], we construct a trial functional in the form

$$J = \int_{t_0}^{t_1} \mathrm{d}t \iiint_{\Omega} L\mathrm{d}V + I_{\mathrm{B}}, \tag{31}$$

where I_{B} is the integral boundary, and L is a trial Lagrangian, which is defined as

$$L = \frac{1}{2}\rho\upsilon^{i}\upsilon_i + \frac{1}{2}\rho I_k^{j}\varpi^{ik}\varpi_{ij} - \frac{1}{2}\varepsilon_{ij}C_{\varepsilon\varepsilon}^{ijmn}\varepsilon_{mn} - \frac{1}{2}e_{ij}C_{ee}^{ijmn}e_{mn} - \frac{1}{2}\gamma_{ijk}C_{\gamma\gamma}^{ijkmnp}\gamma_{mnp} + \frac{1}{2}E_iC_{EE}^{im}E_m + \frac{1}{2}B_iC_{BB}^{im}B_m + f. \tag{32}$$

Here, $p^i, \pi^{ij}, \sigma^{ij}, s^{ij}, m^{kij}, D^i, H^i, v_i, \varepsilon_{ij}, \varpi_{ij}, \gamma_{ijk}, e_{ij}, B_i, E_i, u_i, \varphi_{ij}, A^i$ and ϕ are considered as independent fields, f is an unknown function. There exist alternative approaches for constructing the trial functional [22–26]. Let us introduce an internal energy density A, which is defined as

$$\begin{aligned} A(\varepsilon_{ij}, e_{ij}, \gamma_{ijk}, E_i, B_i) = {} & \frac{1}{2}\varepsilon_{ij}C_{\varepsilon\varepsilon}^{ijmn}\varepsilon_{mn} + \frac{1}{2}e_{ij}C_{ee}^{ijmn}e_{mn} + \frac{1}{2}\gamma_{ijk}C_{\gamma\gamma}^{ijkmnp}\gamma_{mnp} - \frac{1}{2}E_iC_{EE}^{im}E_m \\ & - \frac{1}{2}B_iC_{BB}^{im}B_m + \varepsilon_{ij}C_{\varepsilon e}^{ijmn}e_{mn} + \varepsilon_{ij}C_{\varepsilon\gamma}^{ijmnp}\gamma_{mnp} + e_{ij}C_{e\gamma}^{ijmnp}\gamma_{mnp} - \varepsilon_{ij}C_{\varepsilon E}^{mij}E_m - \varepsilon_{ij}C_{\varepsilon B}^{mij}B_m \\ & - e_{ij}C_{eE}^{mij}E_m - e_{ij}C_{eB}^{mij}B_m - \gamma_{ijk}C_{\gamma E}^{mijk}E_m - \gamma_{ijk}C_{\gamma B}^{mijk}B_m - E_iC_{EB}^{im}B_m. \end{aligned} \tag{33}$$

The Hu–Washizu-like Lagrangian can be written in the form

$$\begin{aligned} L_{\mathrm{HW}} = {} & K - A - p^i(v_i - \dot{u}_i) - \pi^{ij}(\varpi_{ij} - \dot{\varphi}_{ij}) + F^iu_i + L^{ij}\varphi_{ij} + \sigma^{ij}(\varepsilon_{ij} - u_{i;j} + \varphi_{ji}) \\ & + s^{ij}(e_{ij} - \varphi_{ij}) + m^{kij}(\gamma_{ijk} - \varphi_{ij;k}) - D^i(E_i + \phi_{,i}) - H^i(B_i - \varepsilon_{ijk}A^{k;j}) - q\phi - J^iA_i, \end{aligned} \tag{34}$$

where K is the kinetic energy density, which is defined as

$$K = \frac{1}{2}\rho v^iv_i + \frac{1}{2}\rho I_k^j\varpi^{ik}\varpi_{ij}. \tag{35}$$

After incorporating the boundary conditions into the functional, we obtain a determined Hu–Washizu-like action, which reads

$$\begin{aligned} J_{\mathrm{HW}} = {} & \int_{t_0}^{t_1}\mathrm{d}t\iiint_{\Omega}L_{\mathrm{HW}}\mathrm{d}V + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_u}(u_i - \bar{u}_i)\sigma^{ji}n_j\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_m}\bar{M}^{ij}\varphi_{ij}\mathrm{d}S \\ & + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_\varphi}(\varphi_{ij} - \bar{\varphi}_{ij})m^{kij}n_k\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_\sigma}\bar{T}^iu_i\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_\phi}(\phi - \bar{\phi})D^in_i\mathrm{d}S \\ & + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_D}\bar{d}\phi\mathrm{d}S - \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_A}H^i\varepsilon_{ijk}n^j(A^k - \bar{A}^k)\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{\partial\Omega_H}\bar{h}^iA_i\mathrm{d}S \\ & + \iiint_V[-\hat{p}^{i1}u_{i1} + p^{i0}(u_{i0} - \bar{u}_{i0}) + (\bar{p}^{i0} - \hat{p}^{i0})u_{i1} - \hat{\pi}^{ij1}\varphi_{ij1} + \pi^{ij0}(\varphi_{ij0} - \bar{\varphi}_{ij0}) + (\bar{\pi}^{ij0} - \hat{\pi}^{ij0})\varphi_{ij1}]\mathrm{d}V, \end{aligned} \tag{36}$$

where '^' denotes the restricted variation [27]. The action J_{HW} can directly lead to all the fundamental equations (1)–(30).

We can also introduce a complementary internal energy density B, which is defined as

$$\begin{aligned} B(\sigma_{ij}, s_{ij}, m_{ijk}, D_i, H_i) = {} & \frac{1}{2}\sigma_{ij}C_{\sigma\sigma}^{ijmn}\sigma_{mn} + \frac{1}{2}s_{ij}C_{ss}^{ijmn}s_{mn} + \frac{1}{2}m_{ijk}C_{mm}^{ijkmnp}m_{mnp} - \frac{1}{2}D_iC_{DD}^{im}D_m - \frac{1}{2}H_iC_{HH}^{im}H_m \\ & - \sigma_{ij}C_{\sigma s}^{ijmn}s_{mn} - \sigma_{ij}C_{\sigma m}^{ijmnp}m_{mnp} - s_{ij}C_{sm}^{ijmnp}m_{mnp} + \sigma_{ij}C_{\sigma D}^{mij}D_m + \sigma_{ij}C_{\sigma H}^{mij}H_m \\ & + s_{ij}C_{sD}^{mij}D_m + s_{ij}C_{sH}^{mij}H_m + m_{ijk}C_{mD}^{mijk}D_m + m_{ijk}C_{mH}^{mijk}H_m + D_iC_{DH}^{im}H_m. \end{aligned} \tag{37}$$

The Hellinger–Reissner-like Lagrangian can be written in the form

$$\begin{aligned} L_{\mathrm{HR}} = {} & B - K(p^i, \pi^{ik}) + p^i\dot{u}_i + \pi^{ij}\dot{\varphi}_{ij} + F^iu_i + L^{ij}\varphi_{ij} \\ & + \sigma^{ij}(-u_{i;j} + \varphi_{ji}) - s^{ij}\varphi_{ij} - m^{kij}\varphi_{ij;k} - D^i\phi_{,i} + H^i\varepsilon_{ijk}A^{k;j} - q\phi - J^iA_i. \end{aligned} \tag{38}$$

Various variational principles can be deduced from the above obtained functionals. Here we give some Lagrangians for reference:

$$L_0 = K - A + F^iu_i + L^{ij}\varphi_{ij} - q\phi - J^iA_i, \tag{39}$$

$$L_1 = K - A - p^iv_i - \pi^{ij}\varpi_{ij} + s^{ij}e_{ij} + \sigma^{ij}\varepsilon_{ij} + m^{kij}\gamma_{ijk} - D^iE_i - H^iB_i, \tag{40}$$

$$\begin{aligned} L_2 = {} & K - A + (F^i - \dot{p}^i + \sigma^{ij}_{;j})u_i - p^iv_i - \pi^{ij}\varpi_{ij} + (\sigma^{ij} - s^{ij} + m^{kij}_{;k} + L^{ij} - \dot{\pi}^{ij})\varphi_{ij} \\ & + s^{ij}e_{ij} + \sigma^{ij}\varepsilon_{ij} + m^{kij}\gamma_{ijk} + (D^i_{;i} - q)\phi - D^iE_i - H^iB_i - (\varepsilon^{ijk}H_{k;j} + J^i)A_i. \end{aligned} \tag{41}$$

4. Conclusion

A family of generalized variational formations has been proposed for micromorphic magnetoelectroelastodynamics by He's semi-inverse method. The present variational principles will provide a theoretical foundation for numerical techniques for the problem discussed.

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